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CHAPTER 17

# GASPARD MONGE, *GÉOMÉTRIE DESCRIPTIVE*, FIRST EDITION (1795)

## Joël Sakarovitch

On the one hand, descriptive geometry is the culmination of a long and slow evolution of different graphical methods used for representing space. On the other hand, it is the fruit of the fertile imagination of a talented geometrician, heir to the age of enlightenment, committed revolutionary, and brilliant teacher. This ambiguous status between art and science undoubtedly confers to descriptive geometry both its charm and specificity. And if the first goal of Monge was a technical one, Michel Chasles was to consider that the 'New geometry' was born with the Monge lectures.

*First publication.* In *Les Séances des écoles normales recueilliées par des sténographes et revues par des professeurs*, Paris, vol. 1, 49–64 (lecture on 1 pluviose Year III, 1795), 278–285 (9 pluviose), 401–413 (21 pluviose); vol. 2, 149–171 (1 ventose), 338–368 (11 ventose); vol. 3, 61–106 (21 ventose), 332–356 (1 germinal); vol. 4, 87–99 (11 germinal), 291–313 (21 germinal); vol. 7, debates, 28–34 (11 pluviose), 63–74 (16 pluviose), 144–151 (26 pluviose).

*Later editions.* All Paris. 2nd (ed. J.N.P. Hachette), Baudouin, 1799. 132 pages. 3rd (ed. and suppl. Hachette), Courcier, 1811. 4th (ed. B. Brisson), Courcier, 1820. Further reprs., inc. in 2 vols., Gauthier–Villars, 1922.

*New edition* (cited here). In *L'Ecole normale de l'an III, Leçons de mathématiques, Laplace, Lagrange, Monge* (ed. J. Dhombres), Paris: Dunod, 1992, 267–459.

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*Related articles:* Poncelet (§27), von Staudt (§33), Klein (§42).

## 1   INTRODUCTION

When Gaspard Monge (1746–1818) gave his first set of lectures on descriptive geometry in Paris in 1795, no one other than himself had any idea what lay behind this title. In Year III of the revolutionary calendar, Monge succeeded in getting descriptive geometry introduced as a discipline that future teachers would have to study at the new *Ecole Normale*. He also made it the supreme discipline of what was to become the *Ecole Polytechnique* by allotting it half of the lecturing time [Paul, 1980, chs. 2–3]. Yet this discipline was not as new as it might have appeared. Coming out of the first lecture given by his colleague at the *Ecole Normale*, J.L. Lagrange exclaimed, 'I did not know I knew descriptive geometry' [de La Gournerie, 1855, 24].

The best way to find out what descriptive geometry is about is to 'listen' to Monge himself, whose words were carefully recorded by shorthand: 'The purpose of this art is two-fold. First it allows one to represent three-dimensional objects susceptible of being rigorously defined on a two-dimensional drawing. [...] Second [...] by taking the description of such objects to its logical conclusion, we can deduce something about their shape and relative positioning' (Programme of his lectures).

In the prologue to his twelve lectures, which were to be the starting-point of the interest of French mathematicians in geometry and of the upheaval mathematics underwent in the 19th century, Monge defined descriptive geometry as an 'art'. It is a 'science' replied in echo Michel Chasles (1793–1880) in his *Aperçu historique sur l'origine et le développement des méthodes en géométrie* before pursuing word for word with the rest of Monge's definition [Chasles, 1837, 189]. But at the same time, Chasles refused to admit that by itself descriptive geometry had the power to demonstrate fundamental geometrical properties such as whether a curve is planar or not.

This article is dedicated to this 'science' that can demonstrate nothing, or this 'art' that can be said to have provoked an upheaval in mathematics. Indeed, the two visions are not incompatible. As a geometrical method for depicting space, descriptive geometry can be seen both as a graphical technique and as a branch of geometry *per se*. But rather than attempting to place descriptive geometry between art and science, it is perhaps more profitable in the first instance to consider it as a language—which is also what Monge invites us to do. It is 'a language necessary for the engineer to conceive a project, for those who are to manage its execution, and finally for the artists who must create the different components' (Programme). It is, as it were, a language to speak 'space in three dimensions', at least when space is populated with objects 'that can be rigorously defined'.

## 2   GASPARD MONGE

### 2.1   Monge at Mézières

By pure chance in 1764, Monge entered through the back door of one of the most prestigious European engineering schools of the second half of the 18th century, the *Ecole du Génie* at Mézières. Just turned 18 years, his curriculum in a nutshell consisted in brilliant studies in Beaune, his native town, and then at Lyon. During the summer of 1764 he effected a survey of Beaune and drew a plan of it. The school's second in command, who happened to be visiting the town at the time, commended Monge for this work and recruited him to work at Mézières.

Little by little, Monge took over all the science teaching at the *Ecole du Génie*. Beginning as an assistant, he eventually replaced the mathematics professor, the Abbot Bossut, and from 1770 he took charge also of the physics lectures. In addition, he taught drawing, perspective and shadowing, as well as stone cutting. In 1775, he earned himself the title of 'Royal Professor of Mathematics and Physics'.

After having been elected as correspondent of Bossut at the Paris *Académie des Sciences* in 1772, Monge participated in several sessions of the *Académie* and came in contact with the Marquis de Condorcet, A.-A. Lavoisier and A.T. Vandermonde among others. Between 1771 and 1780, he presented eight memoirs, five of which were in analysis (essentially about partial differential equations), and three on differential geometry. Elected 'Associate Geometrician' of the *Académie* in 1780, he left the Mézières school in 1784 and settled in Paris. More interested at that time in physics and chemistry than in mathematics, he actively participated in the studies conducted by chemists in Lavoisier's immediate circle. Indeed, he succeeded in obtaining the synthesis of a small amount of water shortly after Lavoisier.

### 2.2   Monge's pedagogical projects

Monge committed himself body and soul to the revolutionary cause, and his political views were to become more radical in the course of the revolution. The Legislative Assembly elected him Navy Minister immediately after 10 August 1792 (which marks the fall of the monarchy), but he handed in his resignation eight months later. Nevertheless, he continued to participate actively in the revolutionary movement and the Public Welfare Committee's war effort. But his most important 'revolutionary' activity had to do with his participation in the pedagogical debates of the time and their consequences.

Monge was the main architect of the *Ecole Polytechnique,* and the creation of the school represents his most striking participation in the pedagogical projects of the Revolution. The school was destined to become the one training place for military and civil engineers and thus replaced in role the *Ecole du Génie* at Mézières and the *Ecole des Ponts et Chaussées* in Paris, both of which it emulated to a large extent. However, the number of students could not be compared with that of its predecessors under the *ancien régime*: nearly from 400 students were recruited in the first year. The Monge lectures we have been left with and which are discussed below are those that he gave at the *Ecole Normale*. But it is when looking at the way he organized his teaching at the *Ecole Polytechnique* that we can best

Figure 1. Monge, sketched by student L.M.J. Atthalin during a lecture at the *Ecole Polytechnique*, 1802 or 1803 (*Ecole Polytechnique* Archives; photograph by I. Grattan-Guinness).

assess his intentions concerning descriptive geometry. Figure 1 shows him drawn by a student there in about 1803.

Starting on 1 germinal (21 March 1795), Monge gave 34 lectures, that were abruptly interrupted on 7 prairial (26 May). On 8 thermidor (26 July), he resumed his lectures on descriptive geometry as applied to the cutting of wood and stone, perspective and shadowing, all at the fast rhythm of six sessions a 'decade' (the ten-day revolutionary week) until the beginning of year IV (the end of October 1795). After that date, he entrusted his colleague Jean Nicolas Pierre Hachette (1769–1834) with the full responsibility of teaching the descriptive geometry course.

As Monge became more and more involved with the politics of Napoleon, he proportionally disengaged himself from the school. From Napoleon's Italian Campaign in 1797 up until the time that he proclaimed himself Emperor in 1804, Monge was entrusted with a large number of official missions. He accompanied Napoleon in Egypt and became president of the Egyptian Institute. Elected Senator after the coup of 18 brumaire, Napoleon made him Senator of Liège in 1803, and he was to become President of the Senate from 1806 to 1807. But in the period of the Restoration (1815–1816) he was excluded from the *Académie* and died in July 1818.

### 3    THE SUBJECT MATTER OF MONGE'S LECTURES

The contents of Monge's lectures are summarised in Table 1. The lectures begin with a presentation of the conventions used to represent spatial bodies, then continue with a se-

Table 1. Summary by Lectures of Monge's book. The dates are given both in the Revolutionary and normal calendars.

| Dates | Lects. | Pp. | Topics |
|---|---|---|---|
| 1 pluviose – 1 ventose, Year III (20 January – 19 February 1795) | 1–4, pt. 1; and debates | 72 | Programme. Preliminary considerations, problems about straight lines and planes. |
| 1, 11 ventose (19 February, 1 March) | 4, parts 2–5 | 46 | Planes tangent to curved surfaces. Examples of the use of three-dimensional geometry to solve planar geometry problems. |
| 21 ventose (11 March) | 6 | 46 | Intersection of curved surfaces and curves of double curvature. |
| 1 germinal (21 March) | 7 | 25 | Application of surface intersection to the resolution of various problems: sphere inscribed in a pyramid, layout of a point from three sightings. |
| 11, 21 germinal (31 March, 10 April) | 8–9 | 36 | Introduction to differential geometry. |
| 1,11,21 floréal (20, 30 April, 10 May) | 10–12 | | Theory of perspective and shadowing. [Published by Brisson in his 1820 edition, 138–187.] |

ries of solved problems that are sometimes interlaced with more general considerations. The theoretical part of the course is subdivided into five chapters: 'Preliminary considerations', planes tangent to curved surfaces, curves of double curvature, 'The application of surface intersection to the resolution of various problems', and an introduction to differential geometry. In addition, he devoted three lectures to the theory of perspective and shadowing.

### 3.1    *Preliminary considerations*

The first part of the course is very revealing of Monge's general conception of descriptive geometry, for it starts with a lengthy lecture on the possible ways of characterizing a point in space *a priori*. He points out that only two planes are required in order to plot spatial objects as long as one introduces the notion of projection as opposed to that of distance as in analytical geometry. This is where he presents the basic principle of descriptive geometry: given two orthogonal planes, each point in space can be defined in terms of its projections onto these planes. When the two reference planes are folded on top of each other, one obtains on a flat sheet of paper what is known as the 'projected point diagram', that is to say, the two points in the plane that define the point in space (Figure 2).

The projection method indeed allows one to represent polyhedra, the projections of which are entirely determined by the projections of their vertices. But for non-specific surfaces, it is necessary to choose an extra convention and provide the method to construct
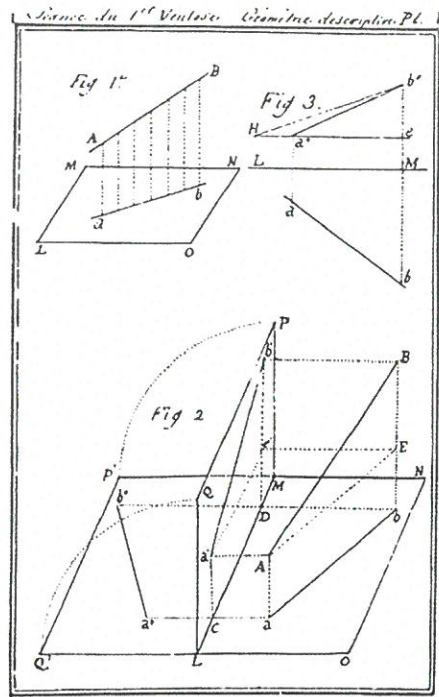
Figure 2. The principle of space representation in descriptive geometry: representation of a point and of a line segment.

the horizontal and vertical projections of two different generators that go through a single point in that surface. Monge gives a few examples of surfaces that can be defined in this way (cylinders, cones and revolution surfaces), and then treats the case of the plane in the same way. He defines the plane like any other surface, the only difference being that the generators that define it, straight lines, are simpler. This order of presentation was never used again in later works.

## 3.2 Tangential surfaces

The first part of the course ends with eight problems about straight lines and planes: tracing the line perpendicular to a given plane and passing through a given point, tracing a plane perpendicular to a given line and passing through a given point, and so on. The second part focuses on tangential planes and the perpendiculars to curved surfaces. Monge naturally begins with the simplest examples: constructing a tangential plane that goes through a single point of the surface of a cylinder, then a cone. Then he surprises his public and the reader by determining the distance between two lines and their common perpendicular. This question is certainly the most interesting problem of elementary descriptive geometry.
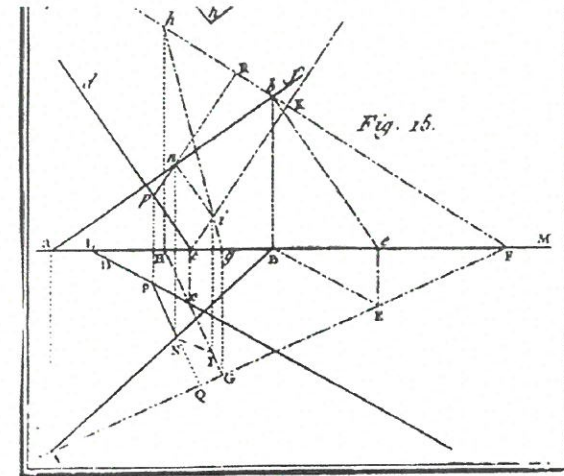
Figure 3. Distance between two straight lines. First, Monge defines the horizontal and frontal tracing of the plane containing the first given straight line (*AB*, *ab*) that is parallel to the second line (*CD*, *cd*). Then he constructs the contact directrix of the cylinder tangent to this plane, the axis of which is the line (*CD*, *cd*). To do this, he defines the projection (*J*, *i*) from any point (Point (*C*, *c*) on the diagram) of this axis onto the plane he has traced. The contact directrix is of course the line—parallel to the axis—that goes through point (*J*, *i*). It cuts line (*AB*, *ab*) at a point that belongs to the common perpendicular, which is therefore completely defined (*PN*, *pn*) since its direction is already known. The true magnitude of the distance between the two lines, which is not traced on the diagram, turns out to be the true length of segment [*PN*, *pn*].

But while it can be resolved by considering only lines and planes and should therefore be approached in the preceding section, Monge treats it in terms of the definition of a revolution cylinder of given axis, tangent to a given plane (parallel to this axis). He presents it, therefore, as the reciprocal of the construction of the plane tangent to a cylinder (Figure 3).

Similarly, this surprising use of auxiliary surfaces allows Monge to deduce two planar geometry theorems from spatial geometry constructs. In the first instance, he demonstrates a theorem of Philippe de La Hire (Figure 4). The principle of Monge's demonstration consists in considering the planar geometry figure as the planar projection of three-dimensional space volumes. A circle is seen as the projection of a sphere, the two tangents to a circle as the generators of a cone. This demonstration, one of the most brilliant examples of the use of three-dimensional geometry to solve a planar geometry problem, brings one directly to the theory of poles and polars, which will be at the heart of work of J.V. Poncelet (1788–1867). Monge generalizes this theorem whilst considering any conical shape. Then, using the same method, he demonstrates the theorem proving (in modern terms) that the homothety centers which change two by two three circles are on a line.
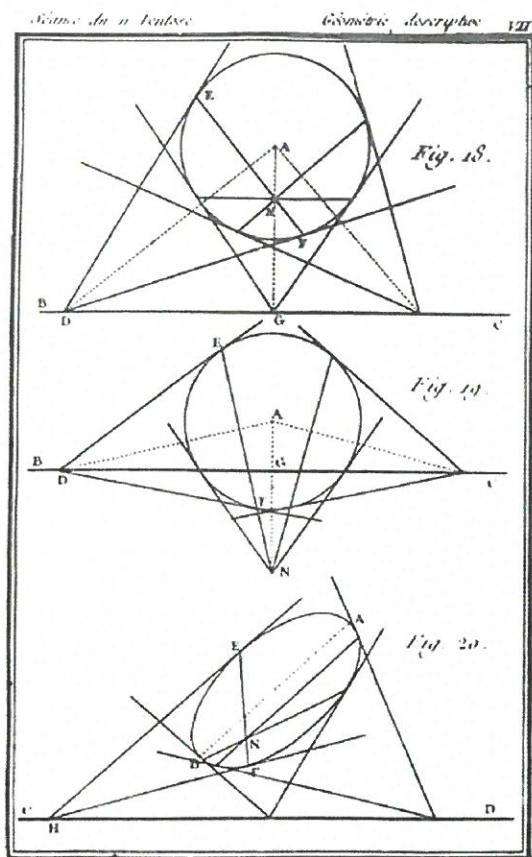
Figure 4. 'Poles and polars'. The chord joining the points where tangents derived from a given point enter into contact with a circle pass through a fixed point when the point moves on a given straight line. Conversely, the tangents derived from the points of intersection of a straight line $\Delta$ and the circle cut one another at a point that moves along a straight line if $\Delta$ turns around a fixed point. Let $\Pi$ be the plane defined by the straight line $\Delta$ and the centre of the circle $A$. Monge considers the sphere centred at point $A$ with the same radius as the circle, and the cones of revolution tangent to the sphere whose vertex moves along the straight line $\Delta$. The cones and the sphere admit the same tangent plane $P$, containing the straight line $\Delta$ ($\Pi$ is a plane of symmetry of the figure and for the rest of the argument we may only consider the volumes situated 'above' $\Pi$). The point $N$ where $P$ comes into contact with the sphere belongs to all the circles of contact between the cones and the sphere; these circles are always situated on the planes perpendicular to $\Pi$. If these volumes are projected onto $\Pi$, the circles of contact are projected on the chords of the circle which pass through N projection of $N$, thus making it possible to deduce the theorem.
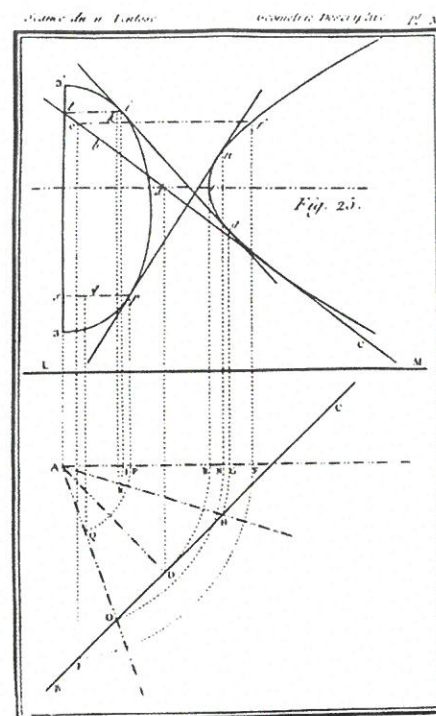


Figure 5. Plane tangent to a surface of revolution passing through a given straight line. Considering the tangent plane he is looking for, Monge supposes it to be rotating according to the motion that generates the surface of revolution. The straight line, included in the plane and labeled $(BC, bc)$ in the Figure, will then generate a rotational hyperboloid. Monge first shows that the plane tangent to the first is also tangent to the second surface of revolution. He then determines point by point the intersection of the rotational hyperboloid with the frontal plane that contains the axis of the first surface of revolution. He finishes off the construction using the tangents common to the hyperbole so defined and the directrix of the first surface of revolution.

Monge ends this section with a far more delicate problem: constructing the plane tangent to a revolution surface passing through a given straight line (Figure 5). This example, like the one concerning the distance between two straight lines, is very revealing about Monge's teaching. It is mainly for him an opportunity to display the gamut of possible auxiliary surfaces, to show that they are not limited to planes, cones and cylinders.

### 3.3   Curves of double curvature and differential geometry

The third part of the course focuses on the intersection of curved surfaces and double curvature curves. Monge takes this opportunity to present the method known as the 'auxiliary planes' method. This consists in having a set of planes intervene, the intersection of these planes with each surface being geometrically defined so that each of the auxiliary planes allows one to construct one (and possibly more) points along the curve of intersection (see Figure 6).

Monge then gives several applications of surface intersection. The two following Lectures, which form the fifth part of the course, do not concern descriptive geometry according to today's nomenclature but some of the results that he had published in some of his memoirs for the *Académie*. In the first of these lectures, appealing to visual and intuitive comprehension, he presents the notion of the evolute of a planar curve as the generalization
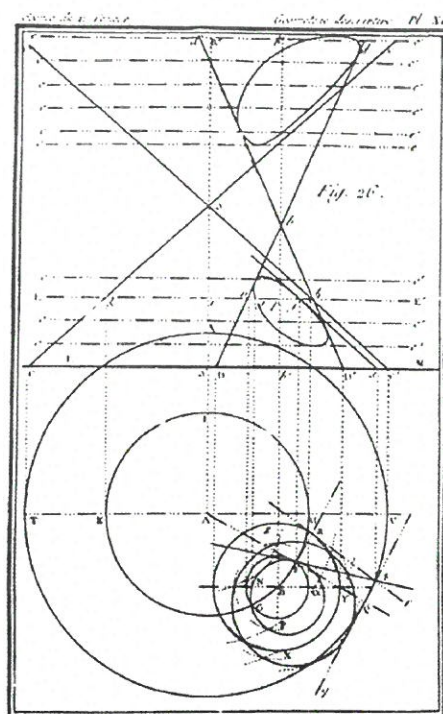
Figure 6. Intersection of two cones; auxiliary plane method.



Figure 7. Evolute and involute of a curve; radius of curvature of a gauche curve; its polar surface.

of a circle, the involute playing the role of the centre (Figure 7). Conversely, the construction of the involute starting with the evolute of a planar curve allows him to introduce the notions of radius of curvature and center of curvature. He defines the polar surface as being the envelope surface of planes normal to the curve (Figure 7). At this point, Monge introduces the notion of developable surface and cuspidal edge. He ends this Lecture by showing that the perpendiculars to a given surface along a curvature line generate a surface that can be developed.

## 3.4   Shadowing and perspective

The theoretical section ends with this complement of differential geometry. However, it does not end the course on descriptive geometry as a whole.

At the *Ecole Normale*, Monge ended up only giving the lectures on shadowing and perspective. But at the *Ecole Polytechnique*, he also taught the applications of descriptive geometry to stone cutting and carpentry, the drawing up of plans and maps, and the technical drawing of machines and for architecture. At the time, descriptive geometry was thus
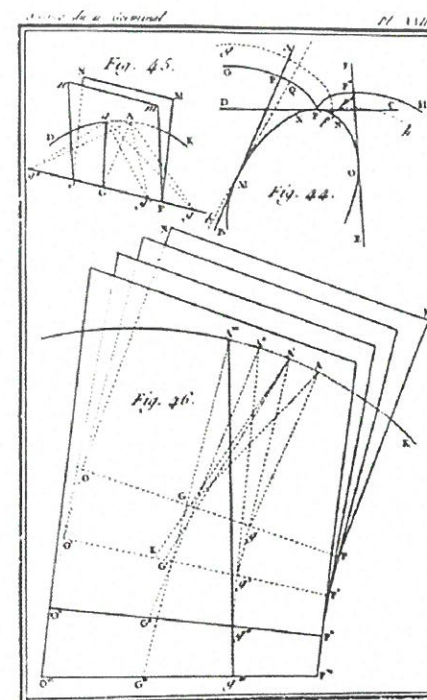
defined in a much broader way than it is today, covering a very large number of subjects (see Figures 8 and 9 as examples).

## 4   THE PRINCIPAL AIMS OF MONGE'S COURSE

### 4.1   Descriptive and practical geometry

Monge never presented descriptive geometry as a new science of which he might be the founding father. Quite the contrary, he describes it as 'having been practiced for a great deal longer [than Analysis] and by many more people'. He even adds that descriptive geometry having been practiced 'by men whose time was precious, the (graphical) procedures were simplified and, instead of considering three planes, one got—thanks to projections—to only require two planes explicitly' (p. 312). Thus, contrary to what is later going to be said against Monge: the minimalist character of the diagram lines used in descriptive geometry is not the fruit of a mathematician's theoretical research but stems from the perfecting of practices over the years. Although he does not cite any names, he is obviously referring to
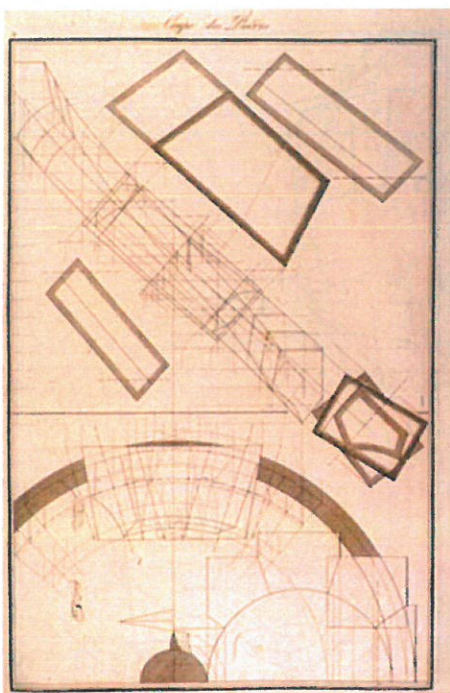
Figure 8. The see-through Saint-Gilles 'Vis', or spiral staircase. Etched diagram from the *Ecole Polytechnique* Archives, files for year III.



Figure 9. Architectural drawing. From the drawings portfolio of student J.-B.-C. Dalesme (1812 promotion), *Ecole Polytechnique* Archives.

the drawings of stone-cutters and carpenters. The privileged ties that descriptive geometry enjoys with various graphical techniques is made evident by the abundant examples that he gives in the foundation course, which is constantly enriched by references to diverse techniques that are likely to use descriptive geometry.

## 4.2   Descriptive geometry and analysis

Monge also returns on several occasions in his lectures to the analogies that exist between descriptive geometry and analysis. He already touches upon this theme in the second Lecture: 'it is not without reason that we are comparing here descriptive geometry and algebra; the two sciences are very closely related. There is no descriptive geometric construct that cannot be transposed in terms of Analysis; and when the problem does not involve more than three unknowns, each analytical operation can be regarded as the script of a play on the geometrical stage' (p. 317).

Monge draws the logical conclusion from this analogy and focuses on it on several occasions: 'it is desirable that the two sciences be cultivated together: descriptive geometry can bring to the most complicated analytical operations the obviousness that characterizes
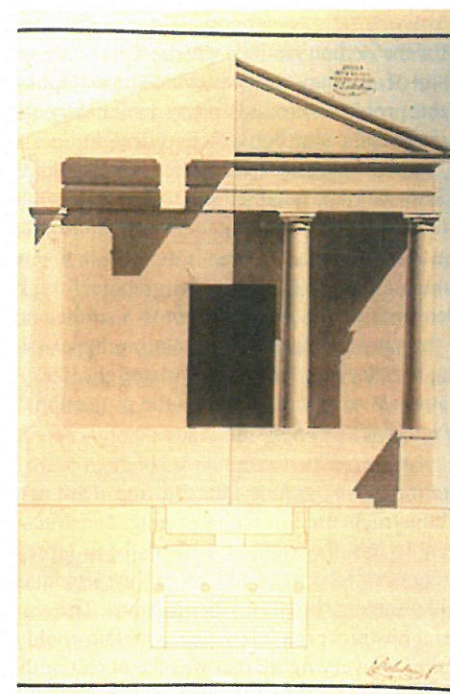
it and, in turn, analysis can bring to geometry the trait of generality which is its essence' (p. 317). In this parallel Monge's philosophy is best expressed. He indeed tried to put it into practice at the *Ecole Polytechnique* where he simultaneously taught descriptive geometry and analysis as applied to geometry (the latter in [Monge, 1795]).

## 4.3   'Properties of surfaces'

'My aim [...] is to get you acquainted with the properties of surfaces', declares Monge on one occasion in a discussion with his students (p. 321). This sentence is probably the best description of this set of lectures. Several elements are brought together to achieve this aim.

First it must be noted that the space imparted in Monge's lectures to the problems of lines and planes is very limited. Descriptive geometry begins with the manipulation of surfaces; it is a tool that allows them to be introduced, conceived, used in proofs and represented.

Faced with the number of solutions that offer themselves, Monge always chooses the most graphic. Cylinders, cones, spheres or other hyperboloids fill the space, providing

matter for the speaker to work from, a support for the listener or the reader's intuition and a certain substance for the demonstrations which, without them, would have been less captivating. The subtle play of auxiliary surfaces, which he manipulates with a consummate art, allows him to turn the problems around and systematically study each problem and its reciprocal. Determining the distance between two straight lines as the reciprocal of the problem of determining the plane tangent to a cylinder is one example. But the most superb illustration of turning the situation around, and the richest from the geometrical point of view, is given in the demonstrations in planar geometry that use descriptive geometry.

Another characteristic element of this course is the way in which Monge expresses the relationship between geometrical reasoning and its graphical translation, its 'representation'. For example, in determining the plane tangent to a surface of revolution (Figure 5), neither the surface nor the tangent plane or the rotating hyperboloid appears explicitly. Similarly, in determining the distance between two straight lines, the cylinder, which is present in the demonstration, is totally absent from the projection diagram (Figure 3); the only element of the surface that has been kept is the one that effectively plays a role, and that is the contact line.

Being pared down as much as possible, the drawing does not show the objects but merely the geometrical constructs used in the reasoning, constructs that would have been drowned and indecipherable had the various surfaces been represented. The projection diagram in descriptive geometry forces one to choose the elements that are needed for the geometric proof. 'The old geometry bristles with diagrams. The reason is simple. Because there was a lack of general abstract principles, each problem could only be analyzed from a concrete standpoint, using the very figure that was the object of the problem. It was only by looking at this figure that one might discover the elements necessary for the proof or the solution one sought', writes Chasles. He even adds, much to the reader's surprise, 'no one has surpassed Monge in conceiving and doing geometry without using figures' [Chasles, 1837, 208]. He points here to one of the riches of Monge's course and highlights the paradoxical contribution of descriptive geometry.

By the very content of his lectures, Monge therefore goes far beyond the narrow and relatively restricting framework he had given himself when, in his introduction, he defined descriptive geometry as a graphical technique. 'And if there is someone amongst you whose [...] heart begins to beat, that is it, he is a geometrician', he declared during one of the debate sessions (p. 321). There is no doubt that his lectures made the heart of many a student beat, and thereby he transformed a whole generation of *Ecole Polytechnique* students into geometers.

## 5    THE INFLUENCE OF MONGE'S LECTURES

### 5.1    *The reputation of descriptive geometry*

The teaching of descriptive geometry developed rapidly. In France, Hachette was to be the most ardent promoter of the Monge theory, which he taught not only at the *Ecole Polytechnique* but also at the Paris *Faculté des Sciences* and at the *Ecole Normale* from 1810 onwards. He also produced new editions of Monge's lectures, a work that was translated into several languages, as the publication history above shows.

In giving a panorama of the history of geometrical methods from antiquity to his time, J.-L. Coolidge treats descriptive geometry with circumspection. While recognizing its technical role, he reduces its scientific value to something of little significance: 'It is hard to point to important properties of space figures which were first found by the methods of Monge or which are more easily proved by those methods than by others' [Coolidge, 1940, 112–113]. This judgement seems rather excessive even if, in the hopes that Monge had placed in the new discipline that he had created, there was something of a revolutionary utopia that was soon to disappear.

Certainly, the theorems on the joining of gauche surfaces or on determining the full shadow separator of the triangular thread screw have neither revolutionized mathematics nor bowled over mathematicians. Nevertheless, Monge's lectures played an important part in the change in mentality that took place at the beginning of the 19th century among mathematicians. They became aware that 'Geometry, which had been looked upon for a century as powerless by itself and having to draw all its resources and acquisitions from algebra, could on the contrary be a source of general principles and methods as fertile as those of algebra, that these methods sometimes had certain advantages in allowing one to penetrate all the way to the origin of truths and lay bare the mysterious chain that links them to each other' [Chasles, 1870, 81].

Three essential ideas appear in Monge's lectures and will be developed afterwards: the notion of projection and transformation, the modification of the relationship between algebra and geometry, and the implicit use of what Poncelet was to call 'the principle of continuity' (§27.1.2). Let us briefly consider them.

### 5.2    *Projections and transformations*

'When thinking carefully about the main advantage of descriptive geometry and the coordinate method, and reflecting upon why these branches of mathematics seem to be akin to absolute doctrines, the principles of which are few but related and linked in a necessary manner and uniform progression, it is not long before one realizes that this is solely due to the use they make of projection' [Poncelet, 1822, 28].

At the heart of descriptive geometry is of course the use of the notion of projection in order to represent points and surfaces from space. But descriptive geometry also allows one to make 'the intimate and systematic link between three-dimensional and planar figures' [Chasles, 1837, 191]. It is in the handling of reciprocal relationships that the true riches of the notions of projection and geometrical transformation become really apparent. C.J. Brianchon, followed by Poncelet, would later successfully cultivate this method, which is one of the hallmarks of the 'Monge School'.

### 5.3    *Geometrical intuition*

The concern and the desire to regain from algebra the terrain that had hitherto escaped geometry are constantly present in the various descriptive geometry treatises and theses of Monge's successors. Felix Klein, who declares 'having been raised [...], thanks to [his] professor, Plücker, in the Monge tradition', considers that one of the major contributions of this tradition was 'the application of geometrical intuition to algebra' (quoted in

[Taton, 1951, 240]). He even adds in *The Erlangen Programme* [Klein, 1872]: 'One must not do away with the prescription that a mathematical problem should not be considered to have been exhaustively examined as long as it has not become intuitively obvious. To discover something by means of algebra is indeed a very important step, but it is only the first step' (compare §42).

## 5.4   The principle of continuity

The 'intimate fusion' [Poncelet, 1822, xx] of two ways of proving a particular property allows one to bring geometrical intuition to the analytical method. But the fact that analytical demonstrations, established in the case of real elements, extend to cases in which some of the elements become imaginary, directly leads Monge to admit that associated geometrical demonstrations must also be extended under the same conditions. In the theorem about poles and polars (Figure 3), there are two distinct cases to be considered *a priori*. The plane tangent to the sphere and including the given straight line only really exists if the line does not intersect with the given circle. Monge indeed traces both figures but makes no distinction in the corpus of the demonstration, apparently taking the fact that the tangent plane might be real or imaginary as negligible and using for the first time the principle of continuity.

### 6   CONCLUSION

Created to 'pull the French nation out of its hitherto dependence on foreign industry' (Programme, 305), descriptive geometry will paradoxically have had more influence in the field of mathematics than in the technical world—contrary to Coolidge's assertion.

Descriptive geometry has been two-faceted from the time it was created. It is on the one hand an entirely new discipline, a 'revolutionary' discipline that acquires a name, and sees its object and place in mathematics defined in Monge's lectures. It offers an unprecedented manner of tackling three-dimensional geometry or, to be more exact, linking planar geometry with spatial geometry. It is also revolutionary because of the position it can aspire to in the school system, in the training of the elite as in general technical training. But it simultaneously appears as the last stage of a tradition that is losing momentum, as the ultimate perfecting of previous graphical techniques and in that capacity, marks the endpoint of an evolutionary process as much as the birth of a new branch of geometry. As such, it can also be viewed as a transition discipline that allowed a gentle evolution to take place: from the 'artist engineer' of the Old Regime, whose training was based on the art of drawing rather than scientific learning, to the 'learned engineer' of the 19th century for whom mathematics—and algebra in particular—is going to become the main pillar of his training.

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2005