Gaspard Monge Founder of "Constructive Geometry"

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ABSTRACT: As a mathematician, or as the "father of the *Ecole polytechnique*", the most famous French engineering school, Gaspard Monge has been the object of numerous studies by historians of science. In this article, I wish to address another aspect of Monge's work, which also allows one to see in Monge the father of "constructive geometry". This field comprises two aspects: either one wishes to determine a surface with construction properties that meet a number of given constraints, or one seeks to build a surface that is given *a priori* in the best possible way. It is the second aspect that Monge developed in his course of descriptive geometry through a "theory of stone assembly". In spite of the imperfections of his theory, Monge inaugurated a highly original way of linking geometry and construction.

Gaspard Monge (1746 – 1818) has gone down in history as a mathematician, the renovator of infinitesimal geometry and founder of descriptive geometry; he has also gone down as the "father of the *Ecole polytechnique*", the most famous French engineering school, founded during the Revolution. In these various respects, he has been the object of numerous studies by historians of science, concerning his role in the renewal of geometry studies in the nineteenth century or his role in the training of the scientific elite (see for example (Taton 1951) or (Belhoste 2003)) .
In this article, I wish to address another aspect of Monge's work, albeit isolated and scantily studied, which also allows one to see in Monge the father of "constructive geometry", which is an original association of geometry and construction. This point incidentally appears in the descriptive geometry founding course that Monge taught in 1795 at the *Ecole normale* of Revolutionary year III and at the *Ecole centrale des travaux publics*, which was to be renamed *Ecole polytechnique* one year later, both having been created in Paris in the heat of the revolutionary years.
I shall therefore give a rapid overview of the conditions in which descriptive geometry emerged and present a "theory of stone assembly", which Monge developed in his course, i.e. the general theory he proposes for dry-stone vaults. It is indeed on this theory that I wish to focus this study, first of all by mentioning its intrinsic limitations and the criticisms it received, before going on to the highly innovative character of Monge's approach, which allows one to grasp, with historical hindsight, another facet of the geometer's fertile mind.

Descriptive geometry at the Ecole polytechnique

When Gaspard Monge gave his first lectures on descriptive geometry in January 1795, no one apart from the orator himself had any idea what the title of this new discipline, born with the institutions where it was being taught, actually encompassed. Presenting the new discipline to his large audience at the *Ecole normale*, he defined it in these terms: "This art has two major objectives. The first is to obtain an exact representation on two-dimensional drawings of three-dimensional objects that require rigorous definition... The second... is to deduce from the exact description of bodies, all that necessarily follows from their shape and their respective positions" (Monge 1795a, pp 305-6).
But from the "two main objectives", history finally remembered the first essentially, letting the second fall into nearly complete oblivion and restricting descriptive geometry to a graphical technique of spatial representation. The example I wish to study here has to do with the "second objective" of descriptive geometry.

I have tried to show in (Sakarovitch 1998) the strong links that exist between the methods and drawings of stone carvers and descriptive geometry. To summarise the situation in a single formula, one might say that stereotomy is to descriptive geometry what perspective is to projective geometry. The parallel nature of their evolution is in fact striking: both practices were developed during the Gothic period, on stone carving sites for the former, in painters' studios for the latter, the first treatises on these matters were published during the Renaissance and their theorisation was formulated at the end of the eighteenth and beginning of the nineteenth century by the mathematicians of the "Monge school". I was thus able to demonstrate the slow transformation of a manual skill into an intellectual skill, which was going to provide the matrix for the geometric operations used in the drawings of foremen, before being formalised as descriptive geometry.
But there is a fundamental difference between them: unlike projective geometry, which breaks away decisively from the graphical techniques from which it sprang forth, descriptive geometry remains linked to the constructive technique from which it was born. Poncelet does not propose a new pictorial art, does not offer advice to painters and has no artistic pretensions. Monge develops a theory of stone assembly on the basis of his geometric theory and this is precisely the point that is of interest to us here.

MONGE'S THEORY OF STONE ASSEMBLY

As part of his course on descriptive geometry, Monge addresses the notion of "lines of surface curvature", which is surprising for the modern reader.
Today, there are three equivalent definitions for lines of curvature:

- They are the curves drawn on the surface, that are tangent at each point to one of the principal directions (i.e. one of the directions where the curvature is extremal)
- They are the curves drawn on the surface with zero geodesic torsion.
- They are the curves drawn on the surface such that the associated normal surface is developable.
  The lines of curvature form an orthogonal net everywhere on the surface except the umbilical points where the principal directions are not defined.

This was initially studied by Euler (Euler 1760), who determined the radii of the osculator circles at all the planar sections of a surface S going through a point, and in particular at the sections of the surface through planes orthogonal to S at this point. Meusnier, one of Monge's most brilliant students at the *Ecole du Génie de Mézières*, was to take Euler's results one step further in the only memoir he submitted in 1776 at the Paris Academy of Sciences (Meusnier 1785). Gauss (Gauss 1828) was to pursue Euler's work, after Monge, and gave these notions the form that we use today (cf. Langevin 2002).
Monge's point of view is different and corresponds to the third definition of the lines of curvature given above. He studied the normals to a surface and showed that along two particular families of curves drawn on the surface – the lines of curvature – these normals constitute two families of developable surfaces.
In his course on descriptive geometry, Monge does not go into technical details but seeks, following his usual approach, to have the notion of lines of curvature understood intuitively. First, he gives the example of hatching on an engraving, which must be drawn according to the lines of curvature of the hatched surface in order to render the nature of the surface as closely as possible (Fig. 1).

Figure 1: Hatching fallowing the lines of curvature of the surface Dalesme's portfolio, 1812.
Exercise at the Ecole polytechnique

Then he develops an actual "theory of assembly". Indeed, he shows that the mode of assembly of a vault is completely governed by the surface chosen for its intrados, if one wants the voussoirs to meet four conditions, which he believes to be essential for the correct execution of a mortarless piece of work (Monge 1795a, pp. 418-420):

- Orthogonality of the voussoir joints with the surface of the vault. This condition is necessary for reasons of statics, otherwise "in the action that two adjacent voussoirs exert on one another, an angle smaller than a right angle would be exposed to exploding".
- Orthogonality of the joints of a single voussoir. This condition answers the same problem of statics as the preceding point but also presents an advantage from an aesthetic point of view since it allows the lines “that divide the vault in layers to be perpendicular to those that divide a given layer into voussoirs".
- The surfaces of the voussoir joints must be developable. This is a practical consideration: if such a surface were not a ruled surface, it would not be executable sufficiently rapidly to be economically viable or with sufficient precision to ensure proper contact between the voussoirs. The fact that the surface is also developable allows greater precision with the use of panels for tracing.
- The lines used to divide the surface "must also provide the character of the surface". This condition, which is purely aesthetic, seems to impose itself on the geometer as powerfully as the previous ones. If the stonework is left bare, the joint lines redraw the surface itself on the vault and appropriate conditions (“*convenances*" in Monge’s terminology) demand perfect harmony between the drawing and the physical limit of the intrados.

"Now, no lines on the curved surface can meet all these conditions at once, except the two suites of curvature lines, and they meet them to the full". As Monge notes immediately after his demonstration, in the case of simple surfaces (revolution surfaces, cylinders, etc.), foremen spontaneously conformed to this law. This is not the same, naturally, for more complex surfaces. In his 1775 memoir (published in 1780), Monge had indeed made sure to point out the fundamental difference that exists between ruled surfaces and developable surfaces and noted in passing that “the only author of stone carving known to artists made a mistake when using development for several surfaces, such as those of groined vaults in a round tower with a ramp or levels, which are not developable and are merely ruled non-developable surfaces”.

Limitations of Monge’s theory

Let us say at the onset that the theory presented by Monge is not without defects and that these were rapidly brought to light (for more details, see (Sakarovitch 1995)). Dupin, in his *Essai historique sur les services et les travaux scientifiques de Gaspard Monge*, mentions in a footnote two restrictions to Monge’s theory, in the case of the underfaces of staircases with a ruled non-developable surface and of the penetration of several vaults; now both types of exception cover a large number of vaults (Dupin 1819, p. 245). Careful examination of the twenty stone carving drawings prepared by Monge for his students at the *Ecole polytechnique*, shows that fourteen of them more or less deviate from the general law enunciated in his course. Even the sloping squinch which, according to Eisenman, served to illustrate "the way Citizen Monge applied his beautiful theory on lines of surface curvature to the division of vaults into voussoirs" (Eisenman An IV, p. 440) does not completely respect the stated rule. In spite of Dupin’s note, Leroy continued to teach the theory of stone assembly according to the lines of curvature from 1815 onwards, without modifying the stone cutting drawings which had been done on the quick when the School was created and were merely copied from J.-B. de La Rue’s treatise.
But the true limitations of Monge’s theory appeared when building relatively specific artworks the development of which was linked to the setting up of railway networks in Europe, namely "oblique bridges". When a road must go over a railway and they both follow more or less the same direction, it is not possible to build a bridge with a deck that is perpendicular to the road it spans without imposing a double curve of very small curvature radius owing to the restricted amount of space available. While it is acceptable for a road spanning a waterway (at least as long as land transport was slow), such a situation is to be banned for trains given the risk of derailment. Hence the construction during the first half of the nineteenth century of a large number of oblique bridges that had to deal with the problem of overload and shaking produced by the passage of trains, while being – sometimes substantially - oblique. The building of such bridges out of stone therefore posed tricky problems of stone layout and gave rise to an extensive literature on the subject. As long as the obliqueness remains above 65° approximately, the stone layout for the bridge can be without too much inconvenience like that of a perpendicular barrel, which is not the case when the obliqueness is more important, at an angle below 40°.
In the article that marks the beginning of theoretical studies on the subject, Lefort quotes the entire passage about the stone layout of vaults in Monge’s course in order to observe that "this analysis, so beautifully done... makes almost complete abstraction of the mechanical aspects that dominate the issue. Indeed it is necessary... to direct the surfaces of the joints perpendicularly to the surface of maximal pressure points, which arise when removing the formwork" (Lefort 1839, p. 290). Now this surface, for a vault with an extrados that is parallel to the intrados, is the surface perpendicular to the line of greatest contraction, a line that is not necessarily indistinguishable from the line of smallest surface curvature. The various solutions proposed – helical stone assembly adopted in England, parallel or convergent orthogonal assembly, cycloidal assembly - determine

"layer and joint lines (that) differ more notably from the lines of greatest and smallest curvature than the exclusive consideration of the geometrical aspects of the question had allowed Monge to admit. Let us note, by the way, that the very condition of having joints that are perpendicular to the vault is more geometrical than mechanical, and that the true definition of the joint surface is apparently this: "This surface must be such that the perpendicular at a given point of the joint follows the direction of the resultant of the pressure forces acting at this point" (Graeff, 1867, p. 12). Coming from a civil engineer, Graeff's criticism is both profound and pertinent. Of course, Monge could not have had in mind a problem that arose frequently after his course. However, there seems to be here a restriction to the generality of the solution proposed, which is "entirely satisfactory [only] when the surface of the vault is such that the line of smallest curvature going through each of these points [can] be entirely traced on the intrados above the side walls" (Lefort, 1839, p. 291).

THE BIRTH OF CONSTRUCTIVE GEOMETRY

The intrinsic limitations of Monge's theory are therefore obvious and the virulence of the criticism he was exposed to may explain the fact that this aspect of Monge's work has not been emphasized (including, I must admit, in my own studies).
Yet, with this "theory of stonework", Monge opened a new research pathway between geometry and construction and threw the foundations for what one might call "constructive geometry of surfaces".
This field comprises two aspects:

- Either one wishes to determine a surface with construction properties that meet a number of given constraints, in as much as possible,
- Or one seeks to build a surface that is given *a priori* in the best possible way.

It is therefore the second aspect that Monge developed or rather inaugurated, to be more precise. He approached it very generally from a geometrical standpoint but completely one-track-mindedly from a constructive standpoint, i.e. through a single construction technique: stereotomy.

It must indeed be emphasized that "constructive geometry" cannot be thought about independently from the material or the construction technique used, which Monge's theory demonstrates here. If one seeks to build a surface given *a priori* "in the best possible way", the best possible way is not something absolute that depends only on the geometry of the surface, but rather something related to the construction technique. "Constructive geometry" intrinsically mixes geometry and execution and in this domain, as in architecture generally, there is no geometry without material. The question – which is implicitly posed by Monge, is therefore not so much how to build a surface given *a priori* in the best way possible, but how to build it with mortarless stones.
One may be surprised that Monge combines conceptual innovation with an ancient construction technique on the decline, in an era, it should be noted, when metallic architecture already had its emblematic work in the form of the Coalbroockdale Bridge. But this association can nonetheless be easily explained. First of all, as mentioned earlier in this article, the affinity that exists between descriptive geometry and stereotomy is to be found and expressed here. Furthermore, this construction technique allows one to execute a very large range of vaulted surfaces, as the history of architecture amply demonstrates, though without having systematically explored the realm of the possible. Finally, while stereotomy is very geometrical in its principle, it is also highly analytical, since each voussoir in a vault can be seen as a decomposition of the surface into "infinitely" small elements and, because of this, is well geared towards the application of tools of analytical geometry. Let us note by the way that it is very common – and perhaps in fact a general rule – in the history of techniques not to be "innovative on all fronts" and that the fact that Monge should rely on a well-tried construction technique to introduce a new conceptual approach to construction is after all not at all surprising.

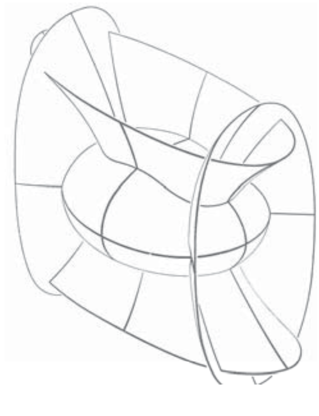



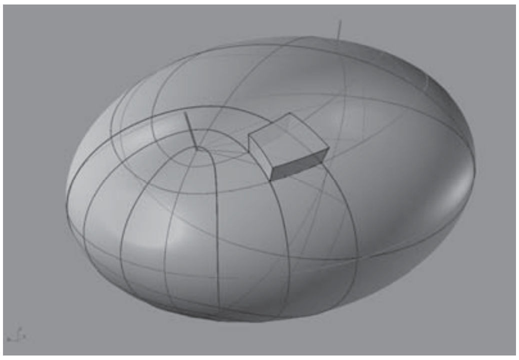
Figure 2 (left): lines of curvature of the ellipsoid Drawing in (Hilbert 1952, p. 189)
Figure 3 (right) : the associated normal surface to the lines of curvature (ibid.)

The only example that Monge really develops in his work on the subject is that of ellipsoid vaults. These vaults, which are common in baroque and classical architecture (they are found for instance in Saint-Peter's Basilica in Rome, in the churches of the Sorbonne or the *Val-de-Grâce* in Paris...), were traditionally assembled, like spherical vaults or ellipsoids of revolution, with horizontal layers. Now Monge had previously studied the ellipsoid and determined its lines of curvature, showing that they are ruled non-developable curves a family of which projects itself horizontally according to a family of ellipses and the other according to a family of hyperbolae (Figs. 2, 3 and 4).

The lecture "on analysis applied to geometry" where Monge deals with this subject has in fact remained in the annals of the *Ecole polytechnique*. On this topic, Arago reports the following anecdote, quoted in (Taton 1951, p. 216): "several professors had eagerly gone to listen to their colleague… At the end of the session, Monge was surrounded and much congratulated. The compliments spoken by Lagrange have been passed on: 'My dear colleague, what you have just exposed was very elegant; I wish I had done it'. Monge admitted having never received a compliment that touched him so deeply".

Ellipsoid vaults therefore provide an excellent example of application of his theory. All the more so since in this particular case, the criticisms mentioned earlier do not apply and the "lines of greatest contraction" and the lines of smallest surface curvature coincide.

Twice in his writings, Monge comes back to the question of ellipsoid vault stone assembly. No doubt in reference to the project for the National Assembly Hall, which was being discussed at the time, he even describes an ideal amphitheatre that might respect the surface geometry (Monge 1795b, Folio n° 20) and (Monge 1796, pp. 162-163). One really feels in this text, fully quoted by Hachette (Hachette 1822, p. 293) and later Leroy (Leroy, 1844, pp. 366-367) in their courses on stereotomy, the wish to see the theory of stone assembly revive mortarless architecture (cf. Sakarovitch 1998, p. 311). One must however observe that this did not happen. We have not been able to find any examples or any mortarless architectural works involving new surfaces or vaults involving intradoses with complex surfaces assembled according to the lines of curvature.



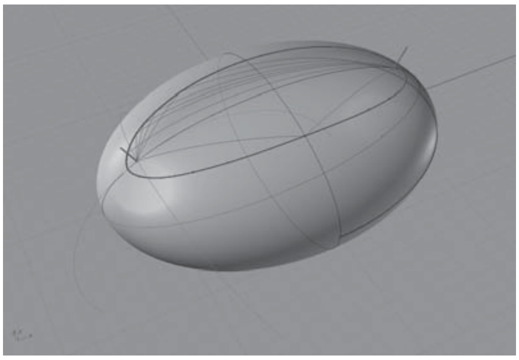


Figure 4: The lines of curvature as generalized ellipses

"The families of the lines of curvature can be regarded as generalized ellipses: choosing a pair of umbilical points not diametrically opposite, we attach the ends of a thread of sufficient length to them and pull it toward a point P of the ellipsoid. The various positions that P can assume on the ellipsoid trace out a line of curvature"; (Hilbert 1952, p. 188).

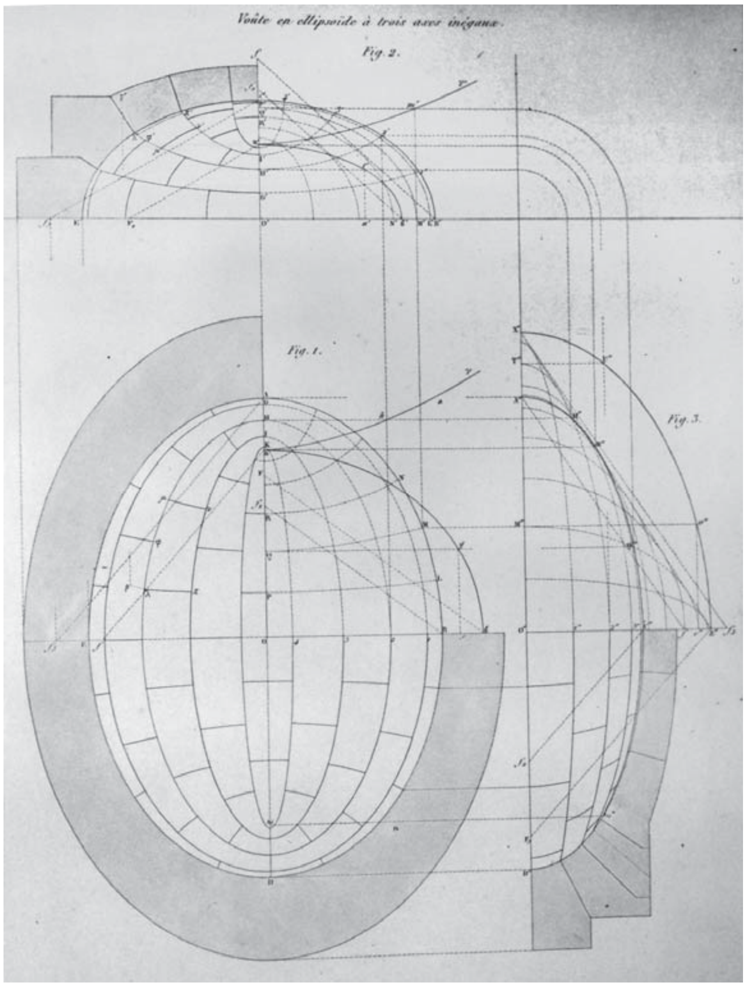


Figure 5 : ellipsoid vault using lines of curvature In (Leroy 1844)

But in spite of what one might consider a double failure – an imperfect theory with no real immediate applications - Monge inaugurates a highly original way of linking geometry and construction, for two reasons:

- First of all, in the way he puts the question: one has known for a long time how to build satisfactorily (from a construction standpoint) a number of vaults with more or less complex intradoses. The question then is what is a good generalisation? Can one find a general answer to this question and therefore a theory of stone assembly? Monge writes, for example, in the *Journal de l'Ecole polytechnique* that "artists almost always excluded from the composition of vaults… the curved surfaces for which they did not know the lines of curvature, even when circumstances demanded them imperatively; and it is mainly to this that we must attribute the poor effect that parts of mortarless vaults often produce in architecture because in order to make a tracing doable, one does not always choose the most appropriate vault surface" (Monge 1796, p. 149).
- Secondly, in the way he answers it, by considering that there is an intrinsic construction logic linked to the surface that must be sought and expressed.

One might even put forward a hypothesis concerning the genesis of Monge's definition of lines of curvature. If the illustration he provides for lines of curvature leads him to a theory of stone assembly, his initial idea may well have been construction-oriented. One might imagine that the question that led him to the definition of lines of curvature was indeed a question concerning the generalisation of the modes of stone vault assembly. No text or testimony is available to confirm this but the definition he gives for lines of curvature from the associated developable normal surfaces does suggest so, as does the fact that he systematically presents these lines of curvature and the theory of stone assembly together.

CONCLUSIONS

Whatever the truth might be, Monge does not pose a purely geometrical problem here, the way Euler did before him or Gauss will pursue after him. Neither is he approaching a pure construction problem consisting in resolving a particular case, like Philibert de l'Orme, who presented the first sloping squinch and exclaimed on this occasion: "this is the first time I have invented something". Monge proposes "the intimate and systematic marriage" of geometry and construction, just as his geometry succeeds in bringing about "the intimate and systematic marriage" of planar geometry and three-dimensional geometry, in the famous words of Michel Chasles.
And even if Monge's approach has not won a following in the area of mortarless architecture, it turned out to be very fertile a century or a century and a half later, in an era of concrete shell structures or light structures.

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